

Optimal Placement Delivery Arrays with Minimum Number of Rows

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Abstract

Coded caching scheme, which is an effective technique to reduce the load during peak traffic times, has recently become quite popular among the coding community. A placement delivery array (PDA in short) can be used to design a coded caching scheme. The number of rows of a PDA corresponds to the subpacketization level in the coded caching scheme. Thus, it is meaningful to construct the optimal PDAs with minimum number of rows. However, no one has yet proved that one of the previously known PDAs (or optimal PDAs) has minimum number of rows. We mainly focus on such optimal PDAs in this paper. We first prove that one class of the optimal PDAs by Maddah-Ali and Niesen has minimum number of rows. Next other two classes of optimal PDAs with minimum number of rows are obtained and proved from a new characterization of a PDA by means of a set of 3 dimensional vectors and a new derived lower bound. Finally three comparisons with previously known results show that our two new optimal PDAs can decrease the number of rows more effectively for some cases.

Index Terms

Coded caching scheme, placement delivery array, lower bound, optimal, minimum.

I. INTRODUCTION

Recently, as the wireless data traffic is increasing at an incredible rate dominated by the video streaming, the wireless network has been imposed a tremendous pressure on the data transmission [1]. Consequently the communication systems are always congested during the peak-traffic times. Caching system, which proactively caches some contents at the network edge during off-peak hours, is a promising solution to reducing congestion (see [2], [6], [7], [9], [12], and references therein).

Maddah-Ali and Niesen in [12] proposed a coded caching approach based on network coding theory. This approach can effectively further reduce congestion during the peak-traffic times, and now is a hot topic in industrial and academic fields. In the coded caching system, [12] focused on the following scenario: a single server containing N files with the same length connects to K users over a shared link and each user has a cache memory of size M files (see Fig. 1). A coded caching scheme

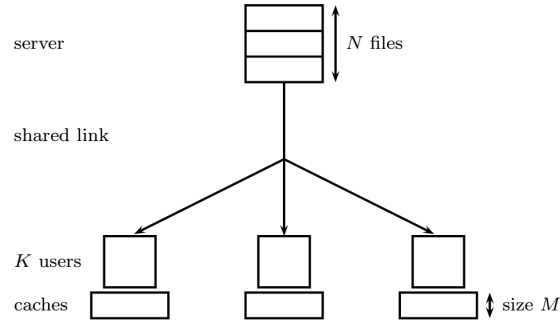


Fig. 1: Coded caching system

consists of two phases: a placement phase during off-peak times and a delivery phase during peak times. In the placement phase, the user caches are populated. This phase does not depend on the user demands which are assumed to be arbitrary. In delivery phase, server sends a coded signal of at most R files to the users such that each user's demand is satisfied. It is meaningful to minimize the load R files in the delivery phase. Here R is always called the delivery rate.

The first determined scheme for a (K, M, N) coded caching system is proposed by Maddah-Ali and Niesen in [12]. Such a scheme is referred to as AN scheme in this paper. According to an elaborate uncoded placement and a coded delivery, the

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(K, M, N) AN scheme is able to reduce the rate R from $K(1 - \frac{M}{N})$ of uncoded caching scheme to $K(1 - \frac{M}{N})\frac{1}{1 + \frac{KM}{N}}$. By means of graph theory, reference [21] showed that AN scheme has minimum delivery rate under the constraint of uncoded cache placement. So far, AN scheme has been also extensively employed in practical scenarios, such as decentralized version [13], device to device networks [8], online caching update [14], [24] and hierarchical networks [10], [27] and so on. So many results have been obtained following [12], for instances, [3], [5], [15], [20], [22], [25] etc.

Nevertheless, there is a central limitation of all previous works, i.e., the problem of *exponential subpacketization*: Coded caching requires every file to be separated into F packets that are then stored in user caches, where F increases exponentially with the number of users K . When K is large, these previous schemes are not practical. So we have to sacrifice some delivery rate for reducing the value of F as much as possible. The first discussion about the problem of subpacketization was proposed by Shanmugam et al. in [17]. Very recently in order to improve exponentially over AN scheme, Yan et al. in [23] characterized (K, M, N) caching system with F subpacketization by a very interesting $F \times K$ array which is called (K, F, Z, S) placement delivery array (PDA), where $M/N = Z/F$ and $R = S/F$. Then they proved that AN scheme is equivalent to a special PDA (AN PDA for short throughout this paper). Furthermore, by increasing little delivery rate, they obtained two infinite classes of PDAs such that F reduces significantly comparing with that of AN PDA. Inspired by the concept of a PDA, there are several discussions on further reducing F by increasing more delivery rate: Shangguan et al. [16] generalized the AN PDAs and the constructions in [23] from the view point of hypergraphs; A class of coded caching schemes with super-polynomial subpacketization from bipartite graphs were proposed by Yan et al. [26]; Based on the results on Ruzsa-Szemerédi graphs, Shanmugam et al. [18] gave a class of coded caching schemes with $K = F$. In fact, all the schemes in [16], [18], [23] can be represented by PDAs. This implies that PDA is an important combinatorial structure to study coded caching system. So the purpose of all the determined coded caching schemes in [16], [17], [18], [23] is to consider the number of rows in a PDA given the parameters K , M/N and R . However, no one has yet proved that one of them above has minimum F . Based on the viewpoint of PDA, another key point should be also considered: a (K, F, Z, S) PDA, which is called optimal in [4], should have minimum value of S given K , F and Z since $R = S/F$. It is worth noting that for fixed parameters K , M/N and R , a PDA with minimum number of rows may not be optimal, and vice versa. Cheng *et al.* [4] constructed several classes of optimal PDAs. Unfortunately no one shows that these PDAs have minimum number of rows. So it is very desirable to construct optimal PDAs with the number of rows as small as possible.

In this paper, we study optimal PDAs with minimum number of rows. Unlike the previously know strategies of constructing PDAs, in this paper we use a different strategy, i.e., we characterize PDAs by means of a set of 3 dimensional vectors. Consequently, several classes of PDAs, two of which are optimal, are obtained. Then a lower bound on the value of S is derived. From the above investigation and lower bound, we prove that 1) AN PDA with $M/N = t/K$ has minimum number of rows for any integers K and $0 < t < K - 1$, and 2) the two new optimal PDAs also have minimum number of rows for the fixed K , $\frac{M}{N}$ and R . Finally three comparisons with previously known results show that our two new optimal PDAs can decrease F more significantly for some parameters K , M/N and R .

The rest of this paper is organized as follows. Section II introduces some preliminaries about PDAs. In Section III, we characterize a PDA by means of a set of 3 dimensional vectors. In Section IV, a relationship among the parameters in a PDA is derived. In Section V three classes of optimal PDAs with minimum F are showed respectively. Finally for some parameters K , M/N and R , three comparisons with previously known results are proposed. Conclusion is drawn in Section VII.

II. PRELIMINARIES

In this paper we denote arrays by bold capital letters, and assume each entry has exactly one symbol. We use $[a, b] = \{a, a + 1, \dots, b\}$ and $[a, b) = \{a, a + 1, \dots, b - 1\}$ for intervals of integers for any integers a and b with $a \leq b$.

Definition 1: ([23]) For positive integers K , and F , an $F \times K$ array $\mathbf{P} = (p_{i,j})$, $i \in [0, F)$, $j \in [0, K)$, composed of a specific symbol “*” called star and S nonnegative integers $0, 1, \dots, S - 1$, is called a (K, F, S) placement delivery array (PDA) if it satisfies the following conditions:

- C1. For any two distinct entries p_{i_1, j_1} and p_{i_2, j_2} , $p_{i_1, j_1} = p_{i_2, j_2} = s$ is an integer only if
 - a. $i_1 \neq i_2$, $j_1 \neq j_2$, i.e., they lie in distinct rows and distinct columns; and
 - b. $p_{i_1, j_2} = p_{i_2, j_1} = *$, i.e., the corresponding 2×2 subarray formed by rows i_1, i_2 and columns j_1, j_2 must be of the following form

$$\begin{pmatrix} s & * \\ * & s \end{pmatrix} \text{ or } \begin{pmatrix} * & s \\ s & * \end{pmatrix}.$$

For any positive integer $Z \leq F$, \mathbf{P} is denoted by (K, F, Z, S) PDA if

- C2. each column has exactly Z stars.

And for any positive integer g , a (K, F, Z, S) PDA is said to be g -regular, denoted by g -(K, F, Z, S) PDA, if each integer of $[0, S)$ appears exactly g times.

Example 1: It is easy to verify that the following array is a 3-(4, 6, 3, 4) PDA:

$$\mathbf{P}_{6 \times 4} = \begin{pmatrix} * & * & 0 & 1 \\ * & 0 & * & 2 \\ * & 1 & 2 & * \\ 0 & * & * & 3 \\ 1 & * & 3 & * \\ 2 & 3 & * & * \end{pmatrix}.$$

In [23], Yan *et al.* showed that a (K, F, Z, S) PDA $\mathbf{P} = (p_{i,j})_{F \times K}$ with $Z/F = M/N$ is corresponding a (K, F, M, N) caching scheme. Precisely, each user is able to decode its requested file exactly for any request with delivery rate $R = S/F$. Thus, it would be preferred to construct a PDA with S as small as possible for given positive integers K, F, Z , and thus we define

$$S(K, F, Z) = \min_{(K, F, Z, S) \text{ PDA}} S \quad (1)$$

Then, a (K, F, Z, S) PDA is said to be optimal if $S = S(K, F, Z)$.

For some fixed parameters $K, M/N$ and R , assume that there exists a (K, F, Z, S) PDA \mathbf{P} with $Z/F = M/N$ and $R = S/F$. We should point that

- \mathbf{P} may not be optimal even if it has minimum F .
- and conversely, the value of F may not be minimum even if \mathbf{P} is optimal.

Example 2: 1) When $K = 6, M/N = 0.5$ and $R = 1.25$, it is easy to check that the following PDA \mathbf{P}' is a $(6, 4, 2, 5)$ PDA with $Z/F = 0.5, S/F = 5/4$. We have that \mathbf{P}' has minimum $F = 4$ since $4|F$. Now we show that \mathbf{P}' is not optimal.

$$\mathbf{P}' = \begin{pmatrix} * & * & * & 0 & 1 & 2 \\ * & 0 & 1 & * & * & 4 \\ 0 & * & 2 & * & 3 & * \\ 1 & 2 & * & 3 & * & * \end{pmatrix}$$

Given $K = 6, F = 4$ and $Z = 2$, we have a $(6, 4, 24)$ PDA $\mathbf{P}_{6 \times 4}^\top$ in reference [4]. Clearly $S = 4$. So \mathbf{P}' is not optimal.

$$\mathbf{P}_{6 \times 4}^\top = \begin{pmatrix} * & * & * & 0 & 1 & 2 \\ * & 0 & 1 & * & * & 3 \\ 0 & * & 2 & * & 3 & * \\ 1 & 2 & * & 3 & * & * \end{pmatrix} \quad (2)$$

2) When $K = 4, F = 4$ and $Z = 2$, \mathbf{P}'' is an optimal $(4, 4, 2, 4)$ PDA by exhaustive computer search. And we have that the following PDA $\mathbf{P}_{2 \times 4}$ has minimum number of rows when $K = 4, M/N = 0.5$ and $R = 1$.

$$\mathbf{P}'' = \begin{pmatrix} * & * & * & 0 \\ * & 0 & 1 & * \\ 0 & * & 2 & * \\ 1 & 2 & * & 3 \end{pmatrix} \quad \mathbf{P}_{2 \times 4} = \begin{pmatrix} 0 & * & 1 & * \\ * & 0 & * & 1 \end{pmatrix}$$

So in the following, we first obtain several classes of optimal PDA, and then focus on the number of rows. For each integer $t \in [0, K)$, let $F = \binom{K}{t}$. Arrange all the subsets with size $t+1$ of $[0, K)$ in the lexicographic order and define $f_{t+1}(\Omega)$ to be its order minus 1 for any subset Ω of size $t+1$. Clearly, f_{t+1} is a bijection from $\{\Omega \subset [0, K) : |\Omega| = t+1\}$ to $[0, \binom{K}{t+1})$. Then, AN PDA is defined as a $\binom{K}{t} \times K$ array $\mathbf{P} = (p_{\mathcal{T}, j})_{\mathcal{T} \subset [0, K), |\mathcal{T}|=t, j \in [0, K)}$ by

$$p_{\mathcal{T}, j} = \begin{cases} f_{t+1}(\mathcal{T} \cup \{j\}) & \text{if } j \notin \mathcal{T} \\ * & \text{otherwise} \end{cases} \quad (3)$$

where the rows are denoted by the sets $\mathcal{T} \subset [0, K)$ and $|\mathcal{T}| = t$ [23].

Example 3: When $K = 4$ and $t = 2$, all the subsets of size $t+1 = 3$ in $\{0, 1, 2, 3\}$ are ordered as $\{0, 1, 2\}, \{0, 1, 3\}, \{0, 2, 3\}$ and $\{1, 2, 3\}$, i.e.,

$$f_3(\{0, 1, 2\}) = 0, \quad f_3(\{0, 1, 3\}) = 1, \quad f_3(\{0, 2, 3\}) = 2 \text{ and } f_3(\{1, 2, 3\}) = 3.$$

Then by (3), we have the $(4, 6, 3, 4)$ PDA $\mathbf{P}_{6 \times 4}$ in Example 1.

It was shown in [21] that the AN PDA has the minimum delivery rate under the constraint of uncoded cache placement. That is, S/F is minimum, which implies $\mathcal{S}(k, \binom{k}{t}, \binom{k-1}{t-1}) = \binom{k}{t+1}$.

Theorem 1: ([23]) The $(t+1)$ - $(k, \binom{k}{t}, \binom{k-1}{t-1}, \binom{k}{t+1})$ PDA with t stars in each row, which is defined in (3), is optimal.

In [4], Cheng *et al.* derived the first lower bound on $\mathcal{S}(K, F, Z)$.

Theorem 2: ([4]) Given any positive integers K, F, Z with $F \geq Z$,

$$\mathcal{S}(K, F, Z) \geq \left\lceil \frac{(F-Z)K}{F} \right\rceil + \left\lceil \frac{F-Z-1}{F-1} \left\lceil \frac{(F-Z)K}{F} \right\rceil \right\rceil + \dots + \left\lceil \frac{1}{Z+1} \left\lceil \frac{2}{Z+2} \left\lceil \dots \left\lceil \frac{(F-Z)K}{F} \right\rceil \dots \right\rceil \right\rceil \right\rceil. \quad (4)$$

According to these lower bounds, the transpose of AN PDA were discussed. For any positive integer k and nonnegative integer t with $t \leq k$, let $\mathbf{P} = (p_{\mathcal{T},j})_{\mathcal{T} \subset [0,k], |\mathcal{T}|=t, j \in [0,k]}$ be a $(k, \binom{k}{t}, \binom{k-1}{t-1}, \binom{k}{t+1})$ PDA. Denote its transpose by \mathbf{P}^\top , i.e.,

Example 4: When $k = 4$ and $t = 2$, a $(6, 4, 2, 4)$ PDA in (2), which is the transpose of the PDA in Example 3, can be obtained.

Theorem 3: ([4]) For any positive integer k and nonnegative integer t with $t \leq k$, the transpose of \mathbf{P} defined in (3) is an optimal $(\binom{k}{t}, k, t, \binom{k}{t+1})$ PDA.

In [4], several infinite classes of optimal PDAs were obtained by using recursive constructions. Particularly, all the optimal PDAs with $Z = 0, 1, F-1$ and F for any positive integers K and F were obtained. In what follows, we always assume that $1 < Z < F-1$ unless otherwise stated.

III. CHARACTERIZATION OF PDA FROM COMBINATORIAL DESIGNS

In this section, we characterize a PDA by means of a set of 3 dimensional vectors. Consequently several classes of PDAs are obtained. For ease of exposition, an $F \times K$ array $\mathbf{P} = (p_{i,j})$, $0 \leq i < F$, $0 \leq j < K$, on $[0, S) \cup \{*\}$ will be represented by a set of ordered triples $\mathcal{C} = \{(i, j, p_{i,j})^T \mid p_{i,j} \in [0, S)\}$ in this section. Clearly an array corresponds to a unique vector set, and the converse is also correct. So we assume that $\mathcal{C} \subseteq [0, F) \times [0, K) \times [0, S)$ and is called the incidence set of \mathbf{P} in the following.

Example 5: The $(4, 6, 3, 4)$ PDA $\mathbf{P}_{6 \times 4}$ in Example 1 can be represented by the following vector set where each column is a vector.

$$\mathcal{C} = \begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 5 \\ 2 & 3 & 1 & 3 & 1 & 2 & 0 & 3 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 & 2 & 0 & 3 & 1 & 3 & 2 & 3 \end{pmatrix}$$

Let \mathbf{x} and \mathbf{y} be elements of \mathcal{C} . The (Hamming) distance between \mathbf{x} and \mathbf{y} , denoted by $d(\mathbf{x}, \mathbf{y})$, is defined to be the number of coordinates at which \mathbf{x} and \mathbf{y} differ. And the (minimum) distance of \mathcal{C} , denoted by $d(\mathcal{C})$, is

$$d(\mathcal{C}) = \min\{d(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in \mathcal{C}, \mathbf{x} \neq \mathbf{y}\}.$$

For instance, $d(\mathcal{C}) = 2$ in Example 5. Furthermore it is easy to check that the following Δ is not a subset of \mathcal{C} in Example 5, where $i_1 \neq i_2 \in [0, 6)$, $j_1 \neq j_2 \in [0, 4)$ and $a \neq b \in [0, 4)$.

$$\Delta = \begin{pmatrix} i_1 & i_1 & i_2 \\ j_1 & j_2 & j_2 \\ a & b & a \end{pmatrix} \quad (5)$$

In fact, we can show that $\Delta \not\subseteq \mathcal{C}$ from C1. We call such Δ a forbidden configuration of \mathcal{C} .

Theorem 4: There exists a (K, F, S) PDA with n integer entries if and only if there exists a set \mathcal{C} with cardinality n satisfying

- P1: the minimum Hamming distance is at least 2, and
- P2: Δ in (5) is the forbidden configuration.

Proof. Given a (K, F, S) PDA \mathbf{P} , denote its incidence set by \mathcal{C} . Clearly the number of integer entries equals the order of \mathcal{C} . Firstly, assume that there exist two distinct vectors, say $(i_1, j_1, a_1)^T$ and $(i_2, j_2, a_2)^T \in \mathcal{C}$, with distance less than 2. It is easy to check that at most one of equalities $i_1 = i_2$ and $j_1 = j_2$ holds since every entry has at most one symbol in \mathbf{P} . Without loss of generality we assume that $i_1 = i_2$. Then we have $a_1 = a_2$. This contradicts C1-a) of Definition 1. So P1 holds. Secondly, suppose that Δ in (5) is a subset of \mathcal{C} . Then we have $p_{i_1, j_1} = p_{i_2, j_2} = a$, $p_{i_1, j_2} = b$, $a, b \in [0, S)$, a contradiction to C1-b) of Definition 1. So P2 holds.

Conversely, assume that a set \mathcal{C} with cardinality n satisfies P1 and P2. From P1, for any integers $i \in [0, F)$ and $j \in [0, K)$, there is at most one integer $a \in [0, S)$ such that $(i, j, a)^T \in \mathcal{C}$. Then we can define an $F \times K$ array $\mathbf{P} = (p_{i,j})$ in the following way:

$$p_{i,j} = \begin{cases} a & \text{if } (i, j, a)^T \in \mathcal{C}, \\ * & \text{otherwise.} \end{cases} \quad (6)$$

Clearly there are n integer entries, and the integer set is $[0, S)$. For any two distinct entries p_{i_1, j_1} and p_{i_2, j_2} , assume that $p_{i_1, j_1} = p_{i_2, j_2} = s \in [0, S)$. Then $i_1 \neq i_2$ and $j_1 \neq j_2$ hold from P1. So C1-a) holds. And $p_{i_1, j_2} = p_{i_2, j_1} = *$. Otherwise, without loss of generality, suppose that $p_{i_1, j_2} \in [0, S)$. Then by (6), there exists a subset $\Delta = \{(i_1, j_1, s)^T, (i_1, j_2, p_{i_1, j_2})^T, (i_2, j_2, s)^T\} \subseteq \mathcal{C}$, a contradiction to P2. So C1-b) holds. \square

From Theorem 4, we can study PDA by means of discussing its incidence set \mathcal{C} . Given a set \mathcal{C} , conjugates of \mathcal{C} are defined by rearranging the coordinates of \mathcal{C} . Let $\mathcal{C}_{(l_0, l_1, l_2)}$ be the conjugate set obtained by rearranging the coordinates of \mathcal{C} in the order $(l_0, l_1, l_2) \in \mathcal{L}$, where \mathcal{L} is the set formed by all the permutations of $[0, 3)$. For instance, \mathcal{C} can be written as $\mathcal{C}_{(0,1,2)}$. $\mathcal{C}_{(2,1,0)}$ is obtained by changing the first coordinate and the third coordinate of \mathcal{C} . It is very easy to verify that the following statement holds.

Lemma 1: \mathcal{C} satisfies P1 and P2 if and only if its conjugates satisfy P1 and P2.

Theorem 5: Let \mathbf{P} be a (K, F, Z, S) PDA for some positive integers K, F, Z and S with $0 < Z < F$. Then

- 1) there exists an $(K, S, S - (F - Z), F)$ PDA;
- 2) if \mathbf{P} is g -regular, then there exist an $(S, F, F - g, K)$ PDA and an $(S, K, K - g, F)$ PDA;
- 3) if each row has h integer entries in \mathbf{P} , then there exist an $(F, S, S - h, K)$ PDA and an $(F, K, K - h, S)$ PDA.

Proof. Let \mathcal{C} be the incidence set of \mathbf{P} . Now let us consider the conjugates of \mathcal{C} .

- When $(l_0, l_1, l_2) = (2, 1, 0)$, $\mathcal{C}_{(2,1,0)} \subseteq [0, S) \times [0, K) \times [0, F)$ satisfies P1 and P2 from Lemma 1. Then $\mathbf{P}_{(2,1,0)}$ generated by $\mathcal{C}_{(2,1,0)}$ in (6) is a (K, S, F) PDA from Theorem 4. It is easy to check that each integer in $[0, K)$ occurs $F - Z$ times in the second coordinate of \mathcal{C} . This implies that each column of $\mathbf{P}_{(2,1,0)}$ has $S - (F - Z)$ stars. So $\mathbf{P}_{(2,1,0)}$ is a $(K, S, S - (F - Z), F)$ PDA.
- When $(l_0, l_1, l_2) = (0, 2, 1)$, $\mathcal{C}_{(0,2,1)} \subseteq [0, F) \times [0, S) \times [0, K)$ satisfies P1 and P2 from Lemma 1. Then $\mathbf{P}_{(0,2,1)}$ generated by $\mathcal{C}_{(0,2,1)}$ in (6) is an (S, F, K) PDA. If \mathbf{P} is g -regular, i.e., each integer, say $s \in [0, S)$ occurs g times in \mathbf{P} , then s occurs g times in the third coordinate of \mathcal{C} . This implies that each column of $\mathbf{P}_{(0,2,1)}$ has $F - g$ stars. So $\mathbf{P}_{(0,2,1)}$ is an $(S, F, F - g, K)$ PDA. Similarly we can show that $\mathbf{P}_{(1,2,0)}$ is an $(S, K, K - g, F)$ PDA.
- When $(l_0, l_1, l_2) = (2, 0, 1)$, $\mathcal{C}_{(2,0,1)} \subseteq [0, S) \times [0, F) \times [0, K)$ satisfies P1 and P2 from Lemma 1. Then $\mathbf{P}_{(2,0,1)}$ generated by $\mathcal{C}_{(2,0,1)}$ in (6) is an (F, S, K) PDA. If each row has h integer entries in \mathbf{P} , i.e., each integer, say $f \in [0, F)$ occurs h times in the first coordinate of \mathcal{C} , then each column of $\mathbf{P}_{(2,0,1)}$ has h integers. That is, each column has $S - h$ stars. So $\mathbf{P}_{(2,0,1)}$ is an $(F, S, S - h, K)$ PDA. Similarly we can show that $\mathbf{P}_{(1,0,2)}$ is an $(F, K, K - h, S)$ PDA.

\square

From Theorems 1 and 5, the following result can be obtained.

Theorem 6: For any positive integers k and t with $0 < t < k - 1$, we have the following PDAs.

- (a) $(t + 1) - (k, \binom{k}{t}, \binom{k-1}{t-1}, \binom{k}{t+1})$ PDA with t stars in each row;
- (b) $(k, \binom{k}{t+1}, \binom{k-1}{t+1}, \binom{k}{t})$ PDA;
- (c) $((\binom{k}{t+1}, \binom{k}{t}) - (t + 1), k)$ PDA;
- (d) $((\binom{k}{t+1}, k, k - (t + 1), \binom{k}{t}))$ PDA;
- (e) $((\binom{k}{t}, \binom{k}{t+1}, \binom{k}{t+1}) - (k - t), k)$ PDA;
- (f) $((\binom{k}{t}, k, t, \binom{k}{t+1}))$ PDA.

For $0 < t' < k - 1$, let $t = k - t' - 1$. Clearly $0 < t < k - 1$. Applying Theorem 6-(a), 6-(c) and 6-(f) to $t = k - t' - 1$, we can obtain $(k, \binom{k}{t+1}, \binom{k-1}{t+1}, \binom{k}{t})$ PDA, $((\binom{k}{t}, \binom{k}{t+1}, \binom{k}{t+1}) - (k - t), k)$ PDA and $((\binom{k}{t+1}, k, k - (t + 1), \binom{k}{t}))$ PDA, i.e., Theorem 6-(b), 6-(d) and 6-(e). So we only need to consider the PDAs in Theorem 6-(a), 6-(c) and 6-(f).

In the practical setting, it is desirable to design caching schemes as well as small F for the fixed $K, \frac{M}{N}$ and R . And this requirement is also proposed in [16] as an open problem. In the following, we will show that the PDAs in Theorem 6-(a), 6-(c) and 6-(f) are optimal and have the minimum F s by the following lower bound on S .

IV. LOWER BOUNDS ON S

In this section, a relationship among the parameters in a PDA is derived. According to this relationship, several classes of optimal PDAs with minimum F will be shown in Section V. Let \mathbf{P} be a (K, F, S) PDA. For any integer $s \in [0, S)$, assume that there are r_s entries in all, say p_{j_u, k_u} , $1 \leq u \leq r_s$, $0 \leq j_u < F$ and $0 \leq k_u < K$, such that $p_{j_u, k_u} = s$. Consider the subarray formed by rows j_1, \dots, j_{r_s} and columns k_1, \dots, k_{r_s} , which is of order $r_s \times r_s$ since $j_u \neq j_v$ and $k_u \neq k_v$ for all $1 \leq u \neq v \leq r_s$ from the definition of a PDA. Further, we have $p_{j_u, k_v} = *$ for all $1 \leq u \neq v \leq r_s$. That is to say, this subarray is equivalent to the following $r_s \times r_s$ array

$$\mathbf{P}^{(s)} = \begin{pmatrix} s & * & \cdots & * \\ * & s & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & s \end{pmatrix} \quad (7)$$

with respect to row/column permutation.

For instance, in Example 1, there are 3 entries $p_{3,0} = p_{1,1} = p_{0,2} = 0$ of $\mathbf{P}_{6 \times 4}$. Then the following subarray formed by rows 0, 1, 3 and columns 2, 1, 0 can be obtained.

$$\mathbf{P}^{(0)} = \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}$$

Theorem 7: If there exists a (K, F, S) PDA \mathbf{P} with $n > 0$ integer entries and integer set $[0, S)$, then

$$\frac{nF}{KF + F - n} \leq S. \quad (8)$$

Further the above equality holds if and only if each row has $\frac{n}{F}$ integer entries and each integer $s \in [0, S)$ occurs exactly $\frac{n}{S}$ times.

Proof. Assume that an integer $s \in [0, S)$ occurs exactly r_s times in \mathbf{P} . From (7), there are $r_s(r_s - 1)$ stars in $\mathbf{P}^{(s)}$. Then the total number of stars in $\mathbf{P}^{(s)}$, $s = 0, 1, \dots, S - 1$, is

$$W = \sum_{s=0}^{S-1} r_s(r_s - 1). \quad (9)$$

On the other hand, we can estimate the value of W in a different way. Without loss of generality, assume that $K < F$, and row i has exactly r'_i integer entries, denoted by $s_{1,i}, \dots, s_{r'_i,i}$. Then each star of i th row occurs in at most r'_i times in $\mathbf{P}^{(s_{1,i})}, \dots, \mathbf{P}^{(s_{r'_i,i})}$ since it occurs in $\mathbf{P}^{(s_{h,i})}$ at most once for any $h \in [1, r'_i]$. So the total number of occurrences of all the stars of \mathbf{P} in $\mathbf{P}^{(s)}$, $s = 0, 1, \dots, S - 1$, is at most

$$W' = \sum_{i=0}^{F-1} r'_i(K - r'_i). \quad (10)$$

Clearly $W \leq W'$, i.e.,

$$\sum_{s=0}^{S-1} r_s(r_s - 1) \leq \sum_{i=0}^{F-1} r'_i(K - r'_i).$$

This implies,

$$\sum_{s=0}^{S-1} r_s^2 + \sum_{i=0}^{F-1} r'^2_i \leq Kn + n. \quad (11)$$

Moreover,

$$\sum_{s=0}^{S-1} r_s^2 \geq \frac{1}{S} \left(\sum_{s=0}^{S-1} r_s \right)^2 = \frac{n^2}{S}, \quad \sum_{i=0}^{F-1} r'^2_i \geq \frac{1}{F} \left(\sum_{i=0}^{F-1} r'_i \right)^2 = \frac{n^2}{F}, \quad (12)$$

where the equalities hold if and only if $r_0 = r_1 = \dots = r_{S-1}$ and $r'_0 = r'_1 = \dots = r'_{F-1}$ respectively. Combining (11) and (12), the inequality (8) can be obtained. And the equality in (8) holds if and only if $r_0 = r_1 = \dots = r_{S-1} = \frac{n}{S}$ and $r'_0 = r'_1 = \dots = r'_{F-1} = \frac{n}{F}$ are positive integers. \square

V. THREE OPTIMAL PDAS WITH MINIMUM NUMBER OF ROWS

Now let us consider the PDAs in Theorem 6-(a), 6-(c) and 6-(f) using the lower bound on the value of S respectively.

A. The first optimal PDA

Lemma 2: Given a (K, F, Z, S) PDA \mathbf{P} , $S = \frac{(F-Z)KF}{Z}$ if and only if \mathbf{P} is a g -regular where $g = KZ/F + 1$.

Proof. Clearly $n = (F - Z)K$. When $S = \frac{(F-Z)KF}{Z}$, i.e., $S = \frac{nF}{KF+F-n}$, from Theorem 7 the number of occurrence of each integer in $[0, S)$ is

$$g = \frac{n}{S} = \frac{K(F-Z)}{\frac{nF}{KF+F-n}} = \frac{KF+F-n}{F} = \frac{KZ}{F} + 1.$$

Conversely if \mathbf{P} is $(\frac{KZ}{F} + 1)$ regular, we have $S(\frac{KZ}{F} + 1) = n$ by counting the number of integer entries. So

$$S = \frac{n}{1 + \frac{KZ}{F}} = \frac{nF}{F + KZ} = \frac{K(F-Z)F}{KF + F - K(F-Z)} = \frac{(F-Z)KF}{Z}.$$

□

Lemma 3: ([23]) For a g -(K, F, Z, S) PDA, if $g = KZ/F + 1$, then $F \geq \binom{K}{KZ/F}$.

From Theorem 7 and Lemma 3, the following result can be obtained.

Lemma 4: For a (K, F, Z, S) PDA with $S = \frac{(F-Z)KF}{Z}$, then $F \geq \binom{K}{KZ/F}$.

So from Lemma 4, the following result holds.

Theorem 8: The optimal $(t+1)$ -($k, \binom{k}{t}, \binom{k-1}{t-1}, \binom{k}{t+1}$) PDA in Theorem 1 has the minimum number of rows for the fixed k and $\frac{M}{N} = \frac{t}{k}$.

B. The second optimal PDA

From Theorem 2 and Theorem 6-(c), the following result can be obtained.

Lemma 5: For any integers k and t with $0 < t < k - 1$, the $((\binom{k}{t+1}, \binom{k}{t}, \binom{k}{t} - (t+1), k)$ PDA \mathbf{P}_1 in Theorem 6-(c) is optimal.

Proof. According to (4)

$$\begin{aligned} & \mathcal{S}\left(\binom{k}{t+1}, \binom{k}{t}, \binom{k}{t} - (t+1)\right) \\ & \geq \left\lfloor \frac{(t+1)\binom{k}{t+1}}{\binom{k}{t}} \right\rfloor + \left\lfloor \frac{t}{\binom{k}{t} - 1} \left\lfloor \frac{(t+1)\binom{k}{t+1}}{\binom{k}{t}} \right\rfloor \right\rfloor + \dots + \left\lfloor \frac{1}{\binom{k}{t} - t} \left\lfloor \frac{2}{\binom{k}{t} - t + 1} \left\lfloor \dots \left\lfloor \frac{(t+1)\binom{k}{t+1}}{\binom{k}{t}} \right\rfloor \dots \right\rfloor \right\rfloor \right\rfloor \\ & = k - t + \left\lfloor \frac{t(k-t)}{\binom{k}{t} - 1} \right\rfloor + \dots + \left\lfloor \frac{1}{\binom{k}{t} - t} \left\lfloor \frac{2}{\binom{k}{t} - t + 1} \left\lfloor \dots \left\lfloor \frac{t(k-t)}{\binom{k}{t} - 1} \right\rfloor \dots \right\rfloor \right\rfloor \right\rfloor \\ & \geq k - t + 1 + \dots + 1 \\ & = k - t + (t+1) - 1 = k. \end{aligned}$$

The forth row of the above formula is derived by the claim $\frac{t(k-t)}{\binom{k}{t}-1} \leq 1$ which can be showed in the following way. It is easy to check that our claim holds when $t = 1$. When $1 < t < k - 1$, it is well know that $\binom{k}{2} \leq \binom{k}{t}$. Then

$$\frac{t(k-t)}{\binom{k}{2} - 1} \leq \frac{t(k-t)}{\binom{k}{2} - \frac{k}{2}} = \frac{2t(k-t)}{k(k-2)} = \begin{cases} \frac{2t}{k} \cdot \frac{k-t}{k-2} & \text{if } 0 < t \leq \lfloor \frac{k}{2} \rfloor, \\ \frac{2(k-t)}{k} \cdot \frac{t}{k-2} & \text{if } \lfloor \frac{k}{2} \rfloor < t < k - 1. \end{cases}$$

This implies that $\frac{t(k-t)}{\binom{k}{2}-1} \leq \frac{t(k-t)}{\binom{k}{2}-\frac{k}{2}} \leq 1$ always holds. So \mathbf{P}_1 is optimal. □

Theorem 9: The optimal $((\binom{k}{t+1}, \binom{k}{t}, \binom{k}{t} - (t+1), k)$ PDA \mathbf{P}_1 in Lemma 5 has the minimum number of rows when $K = \binom{k}{t+1}$, $\frac{M}{N} = 1 - \frac{t+1}{\binom{k}{t}}$ and $R \leq \frac{k}{\binom{k}{t}}$.

Proof. When $K = \binom{k}{t+1}$, $\frac{M}{N} = 1 - \frac{t+1}{\binom{k}{t}}$ and $R = \frac{k}{\binom{k}{t}}$, let us show that \mathbf{P}_1 has the minimum number of rows by contradiction. Assume there exists a $(\binom{k}{t+1}, F', F'(1 - \frac{t+1}{\binom{k}{t}}), K')$ PDA \mathbf{P}' with $F' < \binom{k}{t}$ and $\frac{K'}{F'} \leq \frac{k}{\binom{k}{t}}$. Let $x = F' \frac{t+1}{\binom{k}{t}}$, i.e., $F' = x \frac{\binom{k}{t}}{t+1}$. It is easy to check that the number of integer entries in \mathbf{P}' is

$$n = \left(F' - F' \left(1 - \frac{t+1}{\binom{k}{t}} \right) \right) \binom{k}{t+1} = F'(k-t) = \binom{k}{t+1} x.$$

Let \mathcal{C}' be the incidence set of \mathbf{P}' . Then $\mathcal{C}'_{(0,2,1)} \subseteq [0, F'] \times [0, K'] \times [0, \binom{k}{t+1})$ satisfies P1 and P2 from Lemma 1. So $\mathbf{P}'_{(0,2,1)}$ generated by $\mathcal{C}'_{(0,2,1)}$ in (6) is a $(K', F', \binom{k}{t+1})$ PDA from Theorem 4. From (8) we have

$$\binom{k}{t+1} \geq S' \geq \left\lceil \frac{nF'}{K'F' + F' - n} \right\rceil = \left\lceil \frac{\binom{k}{t+1} x F'}{K'F' + F' - F'(k-t)} \right\rceil \geq \binom{k}{t+1} \frac{x}{K' + 1 - (k-t)}. \quad (13)$$

So we have $1 \geq \frac{x}{K' + 1 - (k-t)}$. Clearly we have $K' + 1 - (k-t) > 0$ since $K'F' + F' - n > 0$. Then the following inequality holds

$$K' - k + (t+1) \geq x \geq 1. \quad (14)$$

From hypothesis

$$\frac{k}{\binom{k}{t}} \geq \frac{K'}{F'} = \frac{K'}{x \frac{\binom{k}{t}}{t+1}}$$

we have

$$\frac{k}{t+1} \geq \frac{K'}{x} \geq \frac{K'}{K' - k + (t+1)}.$$

Then

$$\frac{t+1}{k} \leq \frac{K' - k + (t+1)}{K'}.$$

That is

$$\frac{k-t-1}{K'} \leq \frac{k-t-1}{k}.$$

This implies $K' \geq k$ since $t < k-1$. However it is easy to check that $K' < k$ since $\frac{K'}{F'} \leq \frac{k}{\binom{k}{t}}$, i.e., $\frac{K'}{k} \leq \frac{F'}{\binom{k}{t}} < 1$. So \mathbf{P}_1 has the minimum number of rows when $K = \binom{k}{t+1}$, $\frac{M}{N} = 1 - \frac{t+1}{\binom{k}{t}}$ and $R = \frac{k}{\binom{k}{t}}$. \square

From Theorem 9, in \mathbf{P}_1 we have

$$\frac{M_1}{N_1} = 1 - \frac{t+1}{\binom{k}{t}}, \quad K_1 = \binom{k}{t+1}, \quad F_1 = \binom{k}{t}, \quad R_1 = \frac{k}{\binom{k}{t}}. \quad (15)$$

When $K = \binom{k}{t+1}$, $\frac{M}{N} = 1 - \frac{t+1}{\binom{k}{t}}$, from Theorem 1 we have an optimal PDA $(K, F_{AN}, Z_{AN}, S_{AN})$ PDA \mathbf{P} , where

$$F_{AN} = \binom{\binom{k}{t+1}}{\binom{k}{t+1}(1 - \frac{t+1}{\binom{k}{t}})} = \binom{\binom{k}{t+1}}{k-t} \quad \text{and} \quad S_{AN} = \binom{\binom{k}{t+1}}{\binom{k}{t+1}(1 - \frac{t+1}{\binom{k}{t}}) + 1} = \binom{\binom{k}{t+1}}{k-t-1}.$$

Then

$$R_{AN} = \frac{S_{AN}}{F_{AN}} = \frac{k-t}{\binom{k}{t+1} - k + t + 1}.$$

Now let us show the relationship between the decrease of F and the increase of R based on \mathbf{P} and \mathbf{P}_1 in Theorem 9. From (15), we have

$$\frac{F_1}{F_{AN}} = \frac{\binom{k}{t}}{\binom{\binom{k}{t+1}}{k-t}} \leq \left(\frac{k}{\binom{k}{t+1}} \right)^{k-t} \quad \text{and} \quad \frac{R_1}{R_{AN}} = \frac{k}{t+1} - \frac{k}{\binom{k}{t}} + \frac{k}{(k-t)\binom{k}{t}}. \quad (16)$$

The first item in (16) is derived by the following fact.

$$\begin{aligned}
\frac{\binom{k}{t}}{\binom{k}{k-t}} &= \frac{\frac{k(k-1)\dots(t+1)}{(k-t)!}}{\frac{\binom{k}{t+1}((\binom{k}{t+1}-1)\dots((\binom{k}{t+1}-k+t+1))}{(k-t)!}} = \frac{k(k-1)\dots(t+1)}{\binom{k}{t+1}((\binom{k}{t+1}-1)\dots((\binom{k}{t+1}-k+t+1))} \\
&= \frac{k}{\binom{k}{t+1}} \frac{k-1}{\binom{k}{t+1}-1} \cdots \frac{t+1}{\binom{k}{t+1}-k+t+1} \\
&\leq \left(\frac{k}{\binom{k}{t+1}} \right)^{k-t}
\end{aligned}$$

The last inequality of the above formula holds due to $\frac{k}{\binom{k}{t+1}} \geq \frac{k-x}{\binom{k}{t+1}-x}$ for any positive integer $x \in [1, k-t]$.

Remark 1: Suppose that $K = \binom{k}{t+1}$ and $\frac{M}{N} = 1 - \frac{t+1}{\binom{k}{t}}$. If R increases by a factor of $\frac{k}{t+1}$ times, then F decreases by more than $((\binom{k}{t+1})/k)^{k-t}$ times by (16).

Example 6: Let $t = k - 3$. By (16) we have

$$\frac{F}{F_{AN}} = \frac{8k-16}{((k-1)k-4)((k-1)k-2)} \quad \text{and} \quad \frac{R}{R_{AN}} = \frac{k}{k-2} - \frac{4}{(k-1)(k-2)}.$$

According to above formula, the following table can be obtained.

k	5	6	7	8	9	10	11	12	13	14	15
K	10	20	35	56	84	120	165	220	286	364	455
$\frac{F}{F_{AN}}$	0.194	0.088	0.047	0.028	0.018	0.013	0.009	0.007	0.005	0.004	0.003
$\frac{R}{R_{AN}}$	1.333	1.3	1.267	1.238	1.214	1.194	1.178	1.164	1.152	1.141	1.132

C. The third optimal PDA

Theorem 10: The $((\binom{k}{t}), k, t, (\binom{k}{t+1}))$ PDA \mathbf{P}_2 in Theorem 6-(f) is optimal, and has the minimum number of rows when $K = \binom{k}{t}$, $\frac{M}{N} = \frac{t}{k}$ and $R \leq \frac{\binom{k}{t+1}}{k}$.

Proof. From Theorem 3, \mathbf{P}_2 is optimal. When $K = \binom{k}{t}$, $\frac{M}{N} = \frac{t}{k}$ and $R = \frac{\binom{k}{t+1}}{k}$, let us show that \mathbf{P}_2 has the minimum number of rows by contradiction. Assume there exists a $((\binom{k}{t}), F', F' \frac{t}{k}, S')$ PDA \mathbf{P}' with $F' < k$ and $R' = \frac{S'}{F'} \leq \frac{\binom{k}{t+1}}{k}$. Then $S' \leq \frac{\binom{k}{t+1}}{k} F'$ and \mathbf{P}_2 has $n = F'(1 - \frac{t}{k}) \binom{k}{t}$ integer entries.

Let \mathcal{C}' be the incidence set of \mathbf{P}' . Then $\mathcal{C}'_{(1,0,2)} \subseteq [0, \binom{k}{t}) \times [0, F') \times [0, S')$ satisfies P1 and P2 from Lemma 1. So $\mathbf{P}'_{(1,0,2)}$ generated by $\mathcal{C}'_{(1,0,2)}$ in (6) is a $(F', \binom{k}{t}, S')$ PDA from Theorem 4. Let $K'' = F'$, $F'' = \binom{k}{t}$. From (8) we have

$$\frac{\binom{k}{t+1}}{k} F' \geq S' \geq \left\lceil \frac{nF''}{K''F'' + F'' - n} \right\rceil = \left\lceil \frac{F'(1 - \frac{t}{k}) \binom{k}{t} \binom{k}{t}}{F' \binom{k}{t} + \binom{k}{t} - F'(1 - \frac{t}{k}) \binom{k}{t}} \right\rceil \geq \frac{F'}{1 + F' \frac{t}{k}} \binom{k}{t} \frac{k-t}{k}.$$

So we have

$$\frac{\binom{k}{t+1}}{k} F' \geq \frac{F'}{1 + F' \frac{t}{k}} \binom{k}{t} \frac{k-t}{k}.$$

This implies that $\frac{1}{t+1} \geq \frac{1}{1 + \frac{t}{k} F'}$, i.e., $F' \geq k$. This contradicts our hypothesis. \square

When $K = \binom{k}{t}$ and $\frac{M}{N} = \frac{t}{k}$, from Theorem 1 we have an optimal PDA $(K, F_{AN}, Z_{AN}, S_{AN})$ PDA \mathbf{P} , where

$$F_{AN} = \binom{\binom{k}{t}}{\binom{k}{t} \frac{t}{k}} = \binom{\binom{k}{t}}{\binom{k-1}{t-1}} \quad \text{and} \quad S_{AN} = \binom{\binom{k}{t}}{\binom{k}{t} \frac{t}{k} + 1}.$$

Then

$$R_{AN} = \frac{S_{AN}}{F_{AN}} = \frac{k-t}{t \binom{k}{t} + k} \binom{k}{t}.$$

Now let us show the relationship between the decrease of F and the increase of R based on \mathbf{P} and \mathbf{P}_2 in Theorem 10. From \mathbf{P}_2 , we have $F = k$ and $R = \frac{\binom{k}{t+1}}{k}$. Then

$$\frac{F}{F_{AN}} = \frac{k}{\binom{\binom{k}{t}}{\binom{k-1}{t-1}}} \leq k \left(\frac{t}{k} \right)^{\binom{k-1}{t-1}} \quad \text{and} \quad \frac{R}{R_{AN}} = \frac{\binom{k}{t+1}/k}{\frac{k-t}{t} \binom{k}{t}} = \left(\binom{k-1}{t-1} + 1 \right) \frac{1}{t+1} \quad (17)$$

where the first inequality holds due to $\frac{\binom{k}{t}-x}{\binom{k-1}{t-1}-x} \geq \frac{\binom{k}{t}}{\binom{k-1}{t-1}} = \frac{k}{t}$ for any integer $x \in [1, \binom{k-1}{t-1}]$.

Remark 2: Suppose that $K = \binom{k}{t}$ and $\frac{M}{N} = \frac{t}{k}$. If R increases about $\binom{k-1}{t-1}$ times, then F decreases by more than $\left(\frac{k}{t}\right)^{\binom{k-1}{t-1}}$ times by (17).

Example 7: When $t = 2$, by (17), we have

$$\frac{F}{F_{AN}} = \frac{k}{\binom{\binom{k}{2}}{\binom{k-1}{1}}} = \frac{k}{\binom{k}{2}} \leq \frac{k}{\left(\frac{k}{2}\right)^{k-1}} \quad \text{and} \quad \frac{R}{R_{AN}} = \frac{k}{3},$$

where the first item holds due to $\left(\frac{m}{l}\right)^l \leq \binom{m}{l}$ for any positive integers m and l with $l \leq m$. Then the following table can be obtained.

k	4	5	6	7	8	9	10
$\left(\frac{k}{2}\right)^{k-1}/k$	8	39.1	243	1838.3	16384	168151.3	1953125
$\frac{k}{3}$	1.3	1.7	2.0	2.3	2.7	3.0	3.3

VI. FURTHER PERFORMANCE ANALYSIS

Given the parameters K and M/N , several comparisons with AN PDA were proposed in [16] and [23]. So in this section, the following comparisons between \mathbf{P}_1 , \mathbf{P}_2 and previously known constructions are considered.

- **Comparison 1:** Comparing with PDAs constructed by Yan et al. in [23], the number of rows of \mathbf{P}_1 significantly decrease by increasing a little delivery rate R ;
- **Comparison 2:** For some parameters K and M/N , the performance of \mathbf{P}_1 is better than that of PDAs generalized by Shuangguan et al. in [16];
- **Comparison 3:** The number of packets in a coded caching scheme made by a \mathbf{P}_2 is far less than that of grouping coded caching schemes proposed by Shanmugam et al. in [17] for the same K , M/N and R .

Now let us introduce above three comparisons one by one.

A. Comparison 1

By means of the so-called partitions in [11], [19], Yan et al. constructed the following PDAs.

Lemma 6: ([23]) For any positive integers m and $q \geq 2$, there exist a $(q(m+1), (q-1)q^m, (q-1)^2q^{m-1}, q^m)$ PDA \mathbf{P}_3 .

From Lemma 6, in \mathbf{P}_3 we have

$$\frac{M_3}{N_3} = 1 - \frac{1}{q}, \quad K_3 = q(m+1), \quad F_3 = q^m(q-1), \quad R_3 = \frac{1}{q-1}.$$

From (15), assume that $\frac{M_3}{N_3} = \frac{M_1}{N_1}$ and $K_3 = K_1$, i.e.,

$$1 - \frac{1}{q} = 1 - \frac{t+1}{\binom{k}{t}} \quad \text{and} \quad \binom{k}{t+1} = q(m+1),$$

for some positive integers m , t and k . Then we have

$$m+1 = k-t$$

and

$$\frac{F_1}{F_3} = \frac{\binom{k}{t}}{q^m(q-1)} = \frac{\binom{k}{t}}{\left(\frac{\binom{k}{t}}{t+1}\right)^{k-t-1} \left(\frac{\binom{k}{t}}{t+1} - 1\right)}, \quad \frac{R_1}{R_3} = \frac{(q-1)k}{\binom{k}{t}} = \frac{k}{t+1} - \frac{k}{\binom{k}{t}}. \quad (18)$$

Now let us consider the values of $\frac{F_1}{F_3}$ and $\frac{R_1}{R_3}$ according to t and k .

- When $t = k - 1$, it is easy to check that \mathbf{P}_1 and \mathbf{P}_3 are trivial.
- When $t = k - 2$, (18) can be written in the following way.

$$\frac{F_1}{F_3} = \frac{2(k-1)}{k-2} \quad \frac{R_1}{R_3} = \frac{k-2}{k-1}$$

- When $t \leq k - 3$, (18) can be written in the following way.

$$\frac{F_1}{F_3} \leq \frac{t+1}{k^{k-t-2}(k-1)} \quad \frac{R_1}{R_3} = \frac{k}{t+1} - \frac{k}{\binom{k}{t}} \quad (19)$$

The first above item is derived by the following fact when $3 \leq t \leq k - 3$.

$$\frac{\binom{k}{t}}{t+1} = \frac{k(k-1)\dots(t+1)}{(k-t)!(t+1)} = k \frac{k-1}{t} \frac{k-2}{t-1} \dots \frac{t+3}{3} \frac{t+2}{2} \frac{t+1}{1} \geq k$$

Remark 3: From (19), given K , M/N , then F decreases by more than k^{k-t-2} times when R increases about $\frac{k}{t+1}$ times.

Example 8: When $K_1 = K_3$ and $\frac{M_3}{N_3} = \frac{M_1}{N_1}$, the following table can be obtained for some small positive integers k , t , m and q .

k	t	m	q	F_1/F_3	R_1/R_3
6	2	3	5	0.03	1.6
6	3	2	5	0.2	1.2
7	2	4	7	0.001457	2
7	4	2	7	0.119048	1.2
8	3	4	14	0.000112	1.85714
8	4	3	14	0.001962	1.48571
9	2	6	12	0.000001	2.75
9	6	2	12	0.05303	1.17857
10	2	7	15	0.00000002	3.11111

B. Comparison 2

Based on the result in Lemma 6, a generalized construction was proposed in [16].

Lemma 7: ([16]) There exists an $((\binom{m}{s})q^s, q^m(q-1)^s, (q^m - q^{m-s})(q-1)^s, q^m)$ PDA \mathbf{P}_4 for any positive integers $q \geq 2$, s and m with $s \leq m$.

From Lemma 7, in \mathbf{P}_4 we have

$$\frac{M_4}{N_4} = 1 - \frac{1}{q^s}, \quad K_4 = \binom{m}{s} q^s, \quad F_4 = q^m(q-1)^s, \quad R_4 = \frac{1}{(q-1)^s}.$$

From (15), assume that $\frac{M_1}{N_1} \leq \frac{M_4}{N_4}$ and $K_1 \geq K_4$, i.e.,

$$1 - \frac{t+1}{\binom{k}{t}} \leq 1 - \frac{1}{q^s} \quad \text{and} \quad \binom{k}{t+1} \geq q^s \binom{m}{s}$$

for some positive integers m , t , s and k with $t \leq k$ and $s \leq m$. So we have

$$q^s(t+1) \geq \binom{k}{t} \quad \text{and} \quad \binom{k}{t} \geq q^s \binom{m}{s} \frac{t+1}{k-t}. \quad (20)$$

This implies

$$\left(\frac{\binom{k}{t}}{t+1} \frac{k-t}{\binom{m}{s}} \right)^{1/s} \geq q \geq \left(\frac{\binom{k}{t}}{t+1} \right)^{1/s} \quad \text{and} \quad \binom{m}{s} \leq k-t.$$

Then we have

$$\frac{F_1}{F_4} = \frac{\binom{k}{t}}{q^m(q-1)^s} \leq \frac{q^s(t+1)}{q^m(q-1)^s} = \frac{t+1}{q^{m-s}(q-1)^s} \quad (21)$$

$$\frac{R_1}{R_4} = \frac{k(q-1)^s}{\binom{k}{t}} \leq \frac{k(q-1)^s}{q^s \binom{m}{s} \frac{t+1}{k-t}} = \left(\frac{q-1}{q} \right)^s \frac{k}{t+1} \frac{k-t}{\binom{m}{s}}. \quad (22)$$

With the aid of a computer, we can find out some parameters K , t , m and s listed in the following example satisfying

$$\frac{K_1}{K_4} \geq 1, \quad \frac{M_1}{N_1} \leq \frac{M_4}{N_4}, \quad \frac{F_1}{F_4} < 1 \quad \text{and} \quad \frac{R_1}{R_4} < 1. \quad (23)$$

Example 9: Let $s = m - 1$ in Lemma 7. The following table can be obtained by (21) and (22).

k	t	m	q	K_1/K_4	$\frac{M_1}{N_1}/\frac{M_4}{N_4}$	F_1/F_4	R_1/R_4
7	3	3	3	1.2963	0.9964	0.6481	0.8000
25	22	3	10	7.6667	1	0.2556	0.8804
9	4	4	3	1.1667	0.9973	0.7778	0.5714
13	7	4	6	1.9861	1	0.2648	0.9470
14	9	4	6	2.3171	0.9996	0.3090	0.8741
17	12	4	8	3.0215	0.9999	0.2158	0.9423
20	15	4	10	3.8760	1	0.1723	0.9404
11	5	5	3	1.1407	0.9994	0.9506	0.3810
13	5	5	4	1.0055	0.9992	0.4189	0.8182
13	6	5	4	1.3406	0.9998	0.5586	0.6136
13	7	5	4	1.3406	0.9992	0.5586	0.6136
16	9	5	6	1.7654	0.9999	0.2942	0.8741
18	10	5	8	2.1366	1	0.1908	0.9877
19	12	5	8	2.4604	1	0.2197	0.9054
23	17	5	9	3.0772	1	0.2137	0.9332
25	19	5	10	3.5420	1	0.1968	0.9262
15	7	6	4	1.0474	0.9997	0.5237	0.5664
17	7	6	5	1.0372	0.9999	0.3112	0.8951
17	8	6	5	1.2965	0.9999	0.3890	0.7161
17	9	6	5	1.2965	0.9999	0.3890	0.7161
19	11	6	6	1.6200	1	0.3240	0.7856
20	9	6	7	1.6656	1	0.2379	0.9259
20	10	6	7	1.8321	1	0.2617	0.8418
20	11	6	7	1.6656	1	0.2379	0.9259
21	13	6	7	2.0179	1	0.2883	0.8025
21	14	6	6	2.4923	1	0.4985	0.5644
23	14	6	9	2.3065	1	0.1922	0.9223
23	15	6	8	2.4939	1	0.2672	0.7884
23	16	6	7	2.4311	1	0.3473	0.7295
26	18	6	10	2.6038	1	0.1736	0.9827
28	21	6	9	3.3420	1	0.2785	0.7749

In fact, we can find out several classes of parameters k , t , m and s with $0 < t < k - 1$ and $1 \leq s < m$ such that (23) holds. First we can obtain that

$$q \geq \left(\frac{\binom{k}{t}}{t+1} \right)^{1/s} \geq \left(\frac{k}{k-t} \right)^{\frac{k-t-1}{s}} \quad (24)$$

since

$$\frac{\binom{k}{t}}{t+1} = \frac{\binom{k}{k-t}}{t+1} = \frac{k(k-1)\dots(t+1)}{(k-t)!(t+1)} = \frac{k}{k-t} \frac{k-1}{k-t-1} \dots \frac{t+2}{2} \frac{t+1}{t+1} \geq \left(\frac{k}{k-t} \right)^{k-t-1}$$

The last item is derived by the fact that $\frac{k}{k-t} < \frac{k-x}{k-t-x}$ where $1 \leq x < k-t$.

Moreover we have $k-t-1 \geq s$ since

$$k-t \geq \binom{m}{s} \geq m \geq s+1.$$

This implies that $q > \frac{k}{k-t}$ from (24). Submitting this inequality into (21) and (22), the following results can be obtained.

$$\frac{F_1}{F_4} < (t+1) \left(\frac{k-t}{t} \right)^s \left(\frac{k-t}{k} \right)^{m-s} \quad \frac{R_1}{R_4} < \left(\frac{k-t}{t} \right)^s \frac{k}{t+1} \frac{k-t}{\binom{m}{s}}. \quad (25)$$

Clearly from (20) and (25), for some parameters k , t , m and s , (23) always holds. For instance, when $6 \leq t = k - 6$ and $1 < s < m$, from (25) we have

$$\begin{aligned} \frac{F_1}{F_4} &< (k-5) \left(\frac{6}{k-6}\right)^s \left(\frac{6}{k}\right)^{m-s} = \frac{36(k-5)}{k(k-6)} \left(\frac{6}{k-6}\right)^{s-1} \left(\frac{6}{k}\right)^{m-s-1} \leq \frac{36(k-5)}{k(k-6)} \\ \frac{R_1}{R_4} &< \left(\frac{6}{k-6}\right)^s \frac{k}{k-5} \frac{6}{\binom{m}{s}} < \frac{36k}{(k-5)(k-6)} \left(\frac{6}{k-6}\right)^{s-1} \leq \frac{36k}{(k-5)(k-6)}. \end{aligned}$$

Clearly when $0 < \frac{36(k-5)}{k(k-6)} < \frac{36k}{(k-5)(k-6)} < 1$, i.e., $k > \frac{1}{2}(47 + \sqrt{2089})$, $F_1/F_4 < 1$ and $R_1/R_4 < 1$ in the above formula always hold.

C. Comparison 3

When the number of users K is very large, Shanmugam et al. in [17] proposed a grouping method. That is, for some integers K' and K , assume that $K'|K$ (for the sake of simplicity). First we divide K users into $\frac{K}{K'}$ groups with the same size. And then we use AN PDA to realize a coded caching scheme for each group. Clearly in this method, from Theorem 1 we have

$$F' = \binom{K'}{K'M/N} \quad \text{and} \quad R' = \frac{K'(1 - \frac{M}{N})}{1 + K'\frac{M}{N}} \frac{K}{K'}.$$

For any positive integers k and t , now let us consider the performance of grouping method and \mathbf{P}_2 in Theorem 10. Assume

$$K = \binom{k}{t} \quad \text{and} \quad \frac{M}{N} = \frac{t}{k}.$$

From Theorem 10, we have

$$F_2 = k \quad \text{and} \quad R_2 = \frac{\binom{k}{t+1}}{k} = \frac{k-t}{k(1+t)} \binom{k}{t}.$$

Assume that $R' = R_2$, i.e.,

$$\frac{k-t}{k(1+t)} \binom{k}{t} = \frac{K'(1 - \frac{t}{k})}{1 + K'\frac{t}{k}} \frac{K}{K'} = \frac{k-t}{k + K't} \binom{k}{t}.$$

Then we have $K' = k$. So we have

$$\frac{F_2}{F'} = \frac{k}{\binom{k}{t}}.$$

Remark 4: From the above discussion, for the same parameters $K = \binom{k}{t}$, $\frac{M}{N} = \frac{t}{k}$ and $R = \binom{k}{t+1}/k$, the number of packets in a grouping coded caching scheme proposed in [17] is $\binom{k}{t}/k$ times larger than that of a \mathbf{P}_2 , where k , t are any positive integers with $t < k$.

VII. CONCLUSION

In this paper, a combinatorial characterization of PDA was proposed by means of a set of 3 dimensional vectors. Consequently, several classes of PDAs, two of which are optimal, were obtained. Then a lower bound on the value of S was derived. From the above investigation and lower bound, we proved that 1) AN PDA with $M/N = t/K$ has minimum number of rows any integers K and $0 < t < K - 1$, and 2) the two new optimal PDAs also have minimum number of rows for the fixed K , $\frac{M}{N}$ and R . Finally three comparisons with previously known results show that our two new optimal PDAs can decrease F more effectively for some parameters K , M/N and R .

It would be interesting to characterize and construct more optimal PDAs with minimum number of rows for the other cases.

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